ACADÉMIE DE LA RÉPUBLIQUE SOCIALISTE DE ROUMANIE

REVUE ROUMAINE DE MATHÉMATIQUES PURES ET APPLIQUÉES

TOME XXV, N° 4 1980

TIRAGE À PART

EDITURA ACADEMIEI REPUBLICII SOCIALISTE ROMANIA

Dedicated to Professor
GHEORGHE MARINESCU
on the occasion of his 60th birthday

A NOTE ON WEAKLY COMPACT OPERATORS ON C-*ALGEBRAS

BY

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This paper is intended to be a report on recent results in C^*-algebras, from the point of view of the isomorphic theory of Banach spaces. Most of them originate in the fundamental work of Grothendieck on weakly compact operators and tensor products of Banach spaces. See [5] — [7].

Though the structure of C^*-algebra has been intensively investigated for a long time, the idea to make use of techniques in the isomorphic theory of Banach spaces is quite new.

It was noted by Gordon and Lewis in [4] that $K(H, H)$ (the Banach space of all compact operators acting on the Hilbert space H) and $L(H,H)$ fail to have a local unconditional structure and perhaps the same is true for all non commutative C^*-algebras. However the lack of commutativity is supplied by the presence of a modulus good enough to make possible the use of powerfull techniques originally known for $C(S)$ spaces, especially those developped by H. P. Rosenthal in [12] and [13]. That was the starting point in writting the present report.

The results discussed here have been announced by the author at the Symposium on Functional Analysis and its Applications held at Craiova on November 2—3, 1979.

1. THE DUNFORD -PETTIS PROPERTY

According to [5] a Banach space E is said to have the Dunford-Pettis property if every weakly compact operator defined on E maps weak Cauchy sequences into norm Cauchy sequences. The basic examples are $C(S)$ and $L_1(\mu)$. In the context of C^*-algebras it was noted in [10] the following

1.1. THEOREM (*the noncommutative Dunford-Pettis property*). *If T is a weakly compact operator defined on a C^*-algebra, then*

$$|x_n| \xrightarrow{w} 0 \quad \textit{implies} \quad \|Tx_n\| \to 0.$$

AMS (MOS) subject classification (1980): Primary 46 B 15

Here the modulus is defined by $|z| = \left(\frac{z^*z + zz^*}{2}\right)^{1/2}$.

In the commutative case (i.e., in the case of $C(S)$ spaces) $x_n \xrightarrow{w} 0$ allways implies $|x_n| \xrightarrow{w} 0$ and thus the result above extends Theorem 1 in [5].

1.2. *Example.* The restriction on positivity in Theorem 1.1 above cannot be removed as shows the following construction. Let $(e_n)_n$ be the standard vector basis in l_2 and let $\Phi : K(l_2, l_2) \to K(l_2, l_2)$ be the operator which associates to each compact operator T the operator $T \circ P_1$ where P_k denotes the orthogonal projection onto $\mathbb{C} \cdot e_k$. Then:

i) Φ can be factored through a Hilbert space and thus Φ is weakly compact. In fact, $\Phi = V \circ U$ where $U \in L(K(l_2, l_2), l_2)$ is given by $U(T) = Te_1$ and $V \in L(l_2, K(l_2, l_2))$ is given by $V(x) = \langle \cdot, e_1 \rangle x$.

ii) Let T_n be the compact operator given by $T_n e_k = e_{k+n}$ for $k \leqslant n$ and $T_n e_k = 0$ for $k > n$, $n \in \mathbb{N}$. Then $\langle T_n x, y \rangle \to 0$ for all $x, y \in l_2$, so that $T_n \xrightarrow{w} 0$. Since $|T_n| e_k = e_k$ for $k \leqslant n$ and $|T_n| e_k = 0$ for $k > n$ it is clear that $(|T_n|)_n$ is not weak converging in $K(l_2, l_2)$.

iii) $\Phi(T_n) = T_n \circ P_1 = \langle \cdot, e_1 \rangle e_{1+n}$, so that $\|\Phi(T_m) - \Phi(T_n)\| = \sqrt{2}$ for $m \neq n$. Consequently the image of a weakly converging sequence is not necessarily norm converging.

2. THE DIEUDONNÉ PROPERTY

According to [5] a Banach space E is said to have the Dieudonné property if every operator T defined on E is weakly compact provided that T maps weak Cauchy sequences into weak converging sequences. The basic examples are $C(S)$ spaces and reflexive Banach spaces. In the context of C^*-algebras we can prove results of the following type:

2.1. THEOREM. *Let T be a positive bounded linear mapping from $M = K(H, H)$ into an ordered Banach space E with closed generating cone. Then T is either weakly compact or the restriction of T to a subspace isomorphic to c_0 is an isomorphism.*

Proof. Let us suppose that T is not weakly compact. Then by combining Theorem II.2 (2) in [1] with Lemma 1.1 in [13] it follows that there exists a sequence of norm-1 functionals $x'_n \in E'_+{}^*$, a sequence $(e''_n)_n$ of pairwise disjoint projections in M' and $0 < \varepsilon < \delta$ such that

$$\langle T' x'_n, e''_n \rangle > \delta \text{ and } \sum_{n \neq k} \langle T' x'_k, e''_n \rangle < \varepsilon.$$

Notice that

$$\varphi(x'') = \sup_{\substack{0 \leqslant x \leqslant x'' \\ x \in M}} \varphi(x)$$

* A well known result due to M. G. Krein asserts that the dual cone in E' is generating and thus an appeal to the Baire Category Theorem yields a constant $M > 0$ such that each $y' \in E'$ can be decomposed as follows

$$y' = u' - v' \text{ with } u', v' \geqslant 0 \text{ and } \|u'\|, \|v'\| \leqslant M \|y'\|.$$

for each $x'' \in M''$, $x'' > 0$ and each $\varphi \in E'$, $\varphi \geqslant 0$. Consequently there exists a sequence of norm-1 positive elements $x_n \in M$ such that

$$\langle T'x'_n, x_n \rangle > \delta \text{ and } \sum_{n \neq k} \langle x_n, T'x'_k \rangle < \varepsilon.$$

Given a finite family $(a_n)_n$ of scalars, let us choose an n_0 with $|a_{n_0}| = \sup |a_n|$. Then

$$\|\Sigma a_n T x_n\| \geqslant |a_{n_0} x'_{n_0}(T x_{n_0})| - \sum_{k \neq n_0} |a_k x'_{n_0}(T x_k)| \geqslant$$

$$\geqslant (\delta - \varepsilon)|a_{n_0}| = (\delta - \varepsilon) \sup |a_n|$$

and

$$\|\Sigma a_n T x_n\| \leqslant \|T\| \cdot \|\Sigma a_n x_n\| \leqslant \|T\| \sup |a_n|,$$

which implies that $\overline{\text{Span}}\ (x_n)_n$ is isomorphic to c_0 and the restriction of T to Span $(x_n)_n$ is an isomorphism, q.e.d.

3. THE RADON — NIKODYM PROPERTY

A Banach space E is said to have the Radon-Nikodym property (i.e., E has *R.N.P.*) if every absolutely summing operator from a space $C(S)$ into E is nuclear. The basic examples are reflexive Banach spaces and separable duals. In the last decade much effort was done for understanding the geometry and the isomorphic properties of Banach spaces with R.N.P. The interested reader may consult the monograph of Diestal and Uhl [3] for a complete picture of this field.

In the context of C^*-algebras the author has proved the following characterization which reminds the case of Banach spaces with local unconditional structure treated by him in [8] and [9]:

3.1. THEOREM. *For E a separable C^*-algebra the following assertions are equivalent:*

i) *E' has R.N.P..;*

ii) *E' is weakly compactly generated;*

iii) *E contains no isomorphic copy of l_1;*

iv) *E' contains no isomorphic copy of $L_1[0, 1]$;*

v) *E' contains no isomorphic copy of a space $l_1(\Gamma)$ with* Card $\Gamma > \aleph_0$.

Except for ii) and v) all assertions above are also equivalent for non separable C^*-algebras.

Similar results were obtained by Teleman [14] in the context of elementary C^*-algebras.

Sketch of proof.

The arguments are very similar to those used in [9]. The equivalences *iii)* $\Leftrightarrow$ *iv)* $\Leftrightarrow$ *v)* hold for any Banach space and this fact was remarked by Pelczynski and Hagler. See Hagler and Stegall: *Banach spaces whose duals contain complemented subspaces isomorphic to C* [0, 1]*, Journal of Funct. Anal., **13** (1973), 233—251. It is also well known that $ii) \Rightarrow i \Rightarrow iii)$.

The proof of $iii) \Rightarrow ii)$ depends on the following fact:

Let M_* be a predual of a von Neumann algebra and let A be a closed subspace of M_*. Then either A contains a subspace complemented in M^* and isomorphic to $l_1(\Gamma)$ for some uncountable set Γ, Card $\Gamma > \aleph_0$, or there exists a positive $\varphi \in M_*$ such that

$$x \in M, \ x > 0, \ \varphi(x) = 0 \text{ implies } \lambda(x) = 0$$

for all $\lambda \in A$. Compare with Lemma 1.3 in [12].

Now let us suppose that E contains no isomorphic copy of l_1. Since $iii) \Rightarrow v)$ the above remark yields a positive functional $\varphi \in E'$ such that

$$x \in E'', \ x > 0 \ \ \varphi(x) = 0 \text{ implies } \lambda(x) = 0$$

for all $\lambda \in E'$. By the Radon-Nikodym theorem it follows that $\overline{\text{Span}\,[-\varphi, \varphi]} = E'$ and it remains to observe that the interval $[-\varphi, \varphi]$ is weakly compact.

By the result above it follows that the dual of $K(H_2, H_2)$ i.e., the space of all nuclear operators acting on H_2 has R.N.P.

Added iu proof. After this paper has been accepted for publication we realize that theorem 2 is true for all operators given on $K(H, H)$.

Received November 13, 1979

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